An Intuitive but Not-All-That-Mathematically-Sound Explanation of the Fourier Transform

by Dan Morris

1) Intro

Like many folks out there, I have a pretty good idea what the Fourier Transform is. I believe deep in my heart that a time-domain signal can be represented as a sum of sinusoids. I've written lots of code that depends on computing Fourier Transforms and power spectra, and relied on some library to do the work (mmmmm… Matlab… fft(x)…) .

However, not being a real signal processing type, it was for a long time a mystery why this formula for the Fourier Transform that I'm stealing from Wikipedia:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\, dt$$

…turns a signal into its frequency components. You do some stuff with complex numbers and *e*, then voila, you have a signal turned into its frequency components. In talking with many other engineers who aren't signal processing types, I realized that many people – dare I say *most* people who *use* Fourier Transforms at least occasionally – were in the same boat. If you asked them to *write* the formula for the Fourier Transform (without Wikipedia), they would remember that there must be some *i* and some *e*, but they would be fuzzy on the details. They may have seen a derivation here or there, but it didn't really sink in.

If that's you, I hope I can share with you my basic intuition for the magic, at least enough to help you intelligently understand how to compute a FT, in case you're ever stuck on a lonely island with no Matlab and need to transform something into the frequency domain.

If you don't have any idea what the time and frequency domains are, you should look for a different tutorial. If you are a mathematically rigorous person who is offended by handwaving, you should look for a different tutorial. I will use both proof by example and proof by handwaving.

If you want to skip right to the money and decide whether my casual intuition is suited to you, try going straight to SECTION 4.

Otherwise, here we go… let's start with the least deep and most incomplete explanation ever proposed on what the Fourier Transform does.

2) Least deep and most incomplete explanation ever proposed on what the Fourier Transform does.

Let's say I have some signal x(t):
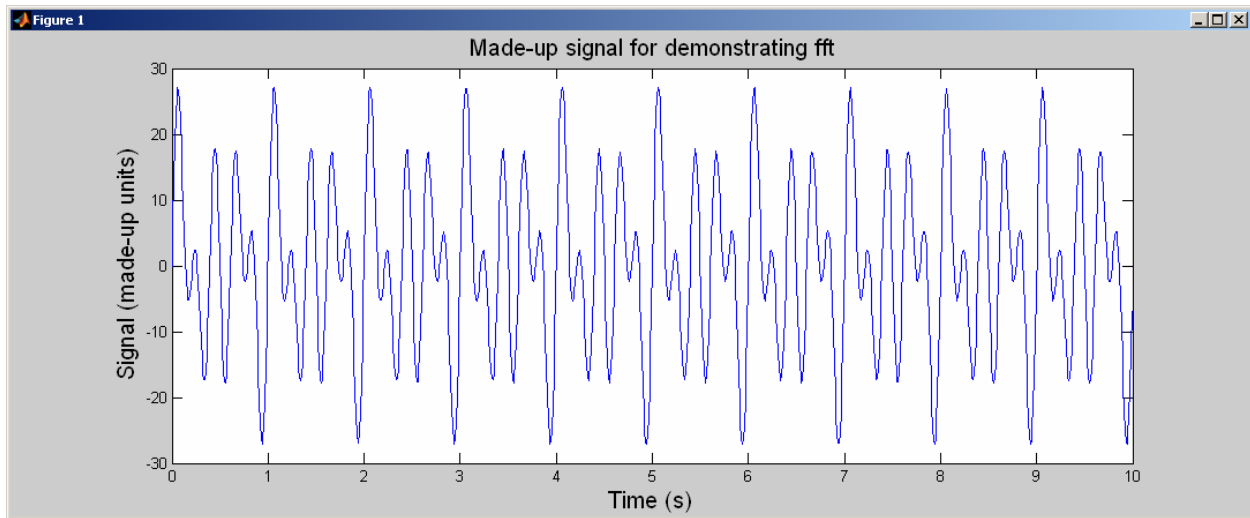
…in this case I made x(t) a bunch of sinusoids added together, to make this more clear. x(t) happens to be three sinusoids with frequencies of 2Hz, 3Hz, and 5Hz and amplitudes 7, 11, and 13 :

$$7\sin(2\pi \bullet 2t) + 11\sin(2\pi \bullet 3t) + 13\sin(2\pi \bullet 5t)$$

I open my favorite library (my personal favorite library is Matlab) and call fft(x) (actually I do a bunch of other stuff to handle Matlab coventions, etc., see the code at the end of this document), which essentially applies the formula I wrote up above, and I get back a magic table telling me "how much of each frequency" makes up my original signal. Clearly in this case, I should get back the three frequencies I put in. What does Matlab show me (code is at the end of this document)?
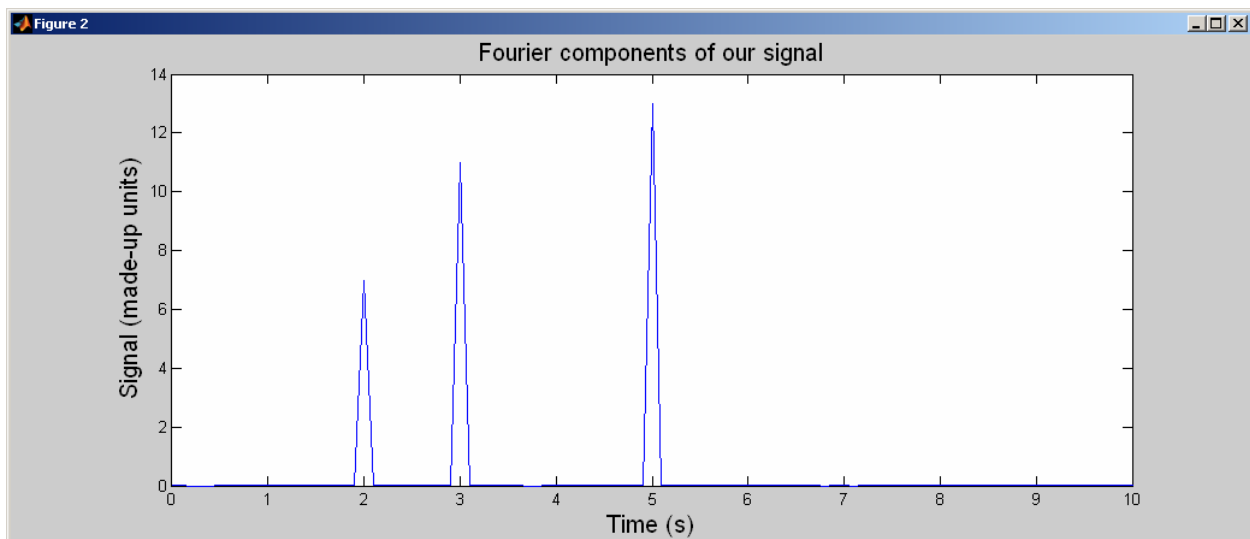
Looks good, I see the exact same three frequencies and amplitudes that I put in.

Great… so this is a pretty good demonstration of the FT's core property: it tells me how much of each frequency "makes up" my original signal. Now again, we ask, how does this formula:

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-i\omega t}\, dt$$

…do all that magic? Let's now jump into the limited mathematical background required for my unsound FT intuition: the complex exponential.

3) Least deep and most incomplete explanation ever proposed on the complex exponential.

The term $e^{i\omega t}$ appears all over the place in signal processing; we call this the complex exponential. What I'm going to do now is try to convince you that this term – $e^{i\omega t}$ – *is* a cosine wave in t where ω determines the frequency, i.e. it's more or less like cos(ωt). While this is an awful simplification that throws out the *complex* part of our exponential, it's really helpful to just sell your brain on this concept, which I'm going to try to do now. If you already believe this and want to move on to the Fourier Transform, skip to Section 4.

First of all there is one thing we have to take on absolute faith, namely that the most magical formula in all of math – Euler's formula – is true:

$e^{i\pi} = $ -1

Most target readers of this document have seen this wizardry before; its derivation is well beyond the scope of this document (and the scope of my head). But I've seen it enough times that I believe it, and hopefully you have too. This is the only thing we're really going to have to take on faith.

Sweet, so what does this have to do with cosine waves? Here's where the (helpful) handwaving begins. Let's look at the complex exponential – $e^{i\omega t}$ – again, more carefully, using some example values. The principle here is going to be that if this function behaves like a cosine wave, then it is a cosine wave. It's a leap, sure, but the point here is intuition, not rigor, so go with me… we're going to plug in some values now and see if $e^{i\omega t}$ looks more or less like a cosine wave.

I'm going to use example values for the term ωt; we'll separate ω and t later. The values I'm going to pick for ωt are the first few multiples of π/2, namely:

0*π/2 , 1*π/2 , 2*π/2 , 3*π/2 , 4*π/2 , and 5*π/2

…otherwise known as:

0, π/2, π, 3π/2, 2π, and 5π/2.

If you don't want to read any math and you trust me not to cheat, you can skip until the </boring algebra> and I'll just show you the results there. I'm just doing algebra here, but it helps me *believe* that this all makes sense.

<boring algebra>

What's the value of the complex exponential when $\omega t = 0$? Well we don't need fancy math for this one, since anything$^0 = 1$:

$e^{i\omega t} = e^{i\,0} = e^0 = 1$.

Okay, what's the value of the complex exponential when $\omega t = \pi/2$? Well, using Euler's formula:

$$e^{i\omega t} = e^{i(\frac{\pi}{2})} = \left(e^{i\pi}\right)^{1/2} = \sqrt{\left(e^{i\pi}\right)} = \sqrt{-1} = i$$

Okay, what's the value of the complex exponential when $\omega t = \pi$? Well, using Euler's formula:

$$e^{i\omega t} = e^{i\pi} = -1$$

Okay, what's the value of this expression when $\omega t = 3\pi/2$? Well, using Euler's formula:

$$e^{i\omega t} = e^{i(\frac{3\pi}{2})} = \left(e^{i\pi}\right)^{3/2} = \left(\left(e^{i\pi}\right)^{\frac{1}{2}}\right)^3 = \left(\sqrt{\left(e^{i\pi}\right)}\right)^3 = \left(\sqrt{-1}\right)^3 = i^3 = i*i*i = -1*i = -i$$

Okay, what's the value of this expression when $\omega t = 2\pi$? Well, using Euler's formula:

$$e^{i\omega t} = e^{i2\pi} = \left(e^{i\pi}\right)^2 = (-1)^2 = 1$$

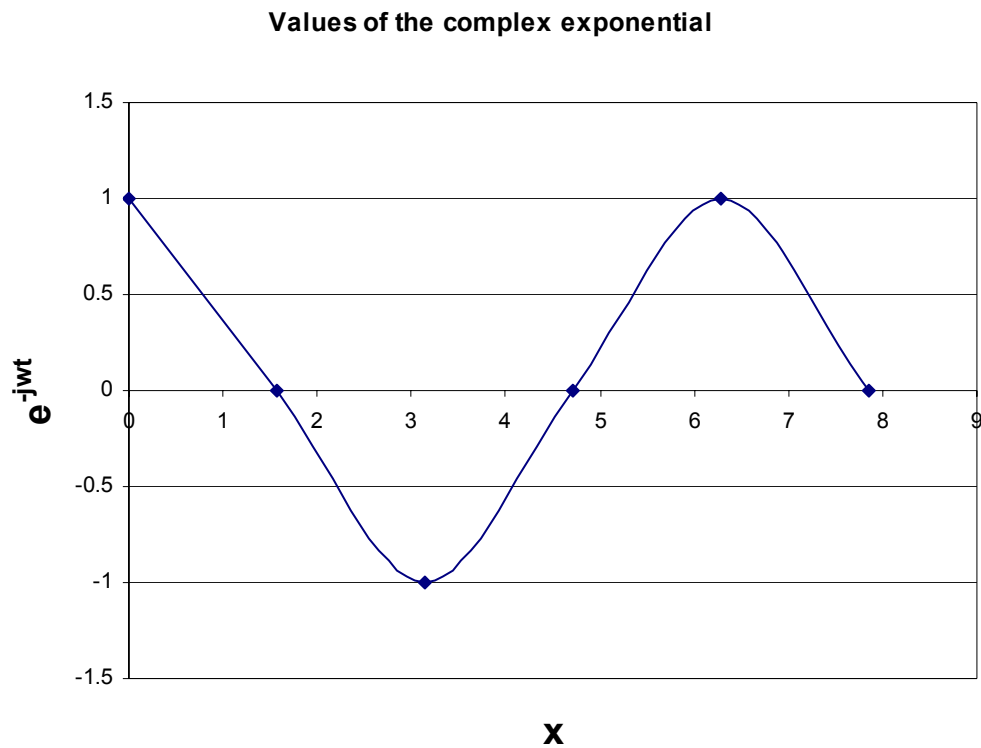Okay, what's the value of the complex exponential when $\omega t = 5\pi/2$? Well, using Euler's formula:

$$e^{i\omega t} = e^{i(\frac{5\pi}{2})} = \left(e^{i\pi}\right)^{5/2} = \left(\left(e^{i\pi}\right)^{\frac{1}{2}}\right)^5 = \left(\sqrt{\left(e^{i\pi}\right)}\right)^5 = \left(\sqrt{-1}\right)^5 = i^5 = i*i*i*i*i = -1*-1*i = i$$

</ boring algebra >

Now let's make a table of those results, and a nice pretty plot:

Value of ωt	Result (value of $e^{i\omega t}$)	Real part of result	Imaginary part of result
0π/2	1	1	0
1π/2	i	0	1*i
2π/2	-1	-1	0
3π/2	-i	0	-1*i
4π/2	1	1	0
5π/2	i	0	1*i

So let's plot the real part of our result (because imaginary numbers are too much to deal with right now):

Values of the complex exponential

Does that plot remind you of anything? Eh? Eh? Maybe it reminds you of cos(x):

cos(x)

x
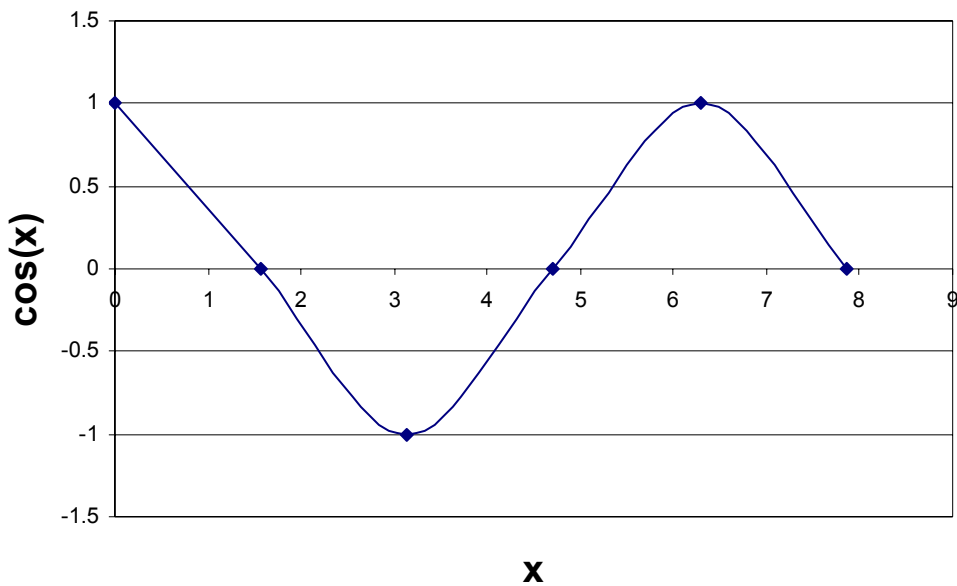
In fact if you plugged all the values you wanted into the complex exponential ($e^{i\omega t}$), the real part of your results would look just like a cosine wave. The complex part, it turns out, would look just like a sine wave at the same frequency! How bloody sweet is that?

So what's up with ω and t, which I tied together? Hopefully you see where I'm going with this… really ω is usually used to represent the frequency of my complex exponential, and t is the input variable. So if e^{it} is the cosine wave we just walked through, e^{2it} is a cosine wave with twice the frequency. I'll leave it as an exercise for you to plug in values and convince yourself of that if you so desire.

So what we have, from all this, what I want you to believe, is that…

$e^{i\omega t}$ is sort of like a cosine wave with frequency ω

…which, by the well-known handwaving theorem, translates to

$e^{i\omega t}$ _is_ a cosine wave with frequency ω

Now we're ready to move on to the Fourier Transform itself.

4) Least deep and most incomplete explanation ever proposed why the Fourier Transform works.

Okay, now equipped with our deep belief that $e^{i\omega t}$ _is_ a cosine wave with frequency ω (if you skipped here from the beginning, I hope you believe that), let's pretend we're otherwise high school algebra students, and engage in a little dialog with a random dude.

Random dude: I need to know how many 7's there are in 49.

High school algebra student: Duh, just divide 49 by 7. There are seven 7's in 49.

Random dude: I need to know how many x's there are in y.

High school algebra student: Duh, just divide y by x. There are y/x x's in y.

Random dude: I need to know how much of the frequency 23Hz there is in my signal x(t).

High school algebra student: Duh, just divide x(t) by 23Hz. There is x(t)/23Hz amount of 23Hz in your signal.

Now this student may not even know what hertz means, but this is a great intuition... sort of... how would we divide x(t) by a signal representing a certain frequency ω (in this case 23Hz)? Well, now that we believe that $e^{i\omega t}$ is a cosine wave with frequency ω, I bet we can use that... how about this:

$$\text{amount of signal with frequency } \omega \text{ in x(t)} = \frac{x(t)}{e^{i\omega t}} = x(t)e^{-i\omega t}$$

This is a nonsense mathematical expression, but it's really getting close to the formula for the Fourier transform... unfortunately, this is an expression that looks at a single value of t. And that doesn't make sense; a single value of x(t) is just a number, and it doesn't have frequency components. What I really want to know is what the amount of signal with frequency ω there is in my *entire* x(t). So what if we integrate this over all values of t?

$$\text{amount of signal with frequency } \omega \text{ in x(t)} = \frac{x(t)}{e^{i\omega t}} = \int_{t=-\infty}^{t=\infty} x(t)e^{-i\omega t}dt$$

Is it... could it be... is that the Fourier Transform? It damn sure looks like the formula at the top of this document that I stole from wikipedia. Let's just finish this up by replacing "amount of signal with frequency ω in x(t)" by the more conventional notation X(ω):

$$X(\omega) = \int_{t=-\infty}^{t=\infty} x(t)e^{-i\omega t}dt$$

So how does this related to the fft(x) function that I like to call all the time in Matlab or some other library? This formula takes the signal x(t) and tells me how much of *one frequency* there is

in that whole signal. The usual implementation of a Fourier Transform does this for lots and lots of frequencies (often you just tell it how many you want), and returns a table of $X(\omega)$ values for different frequencies. This is exactly what the plot is in section 1.

Hopefully that at least helps you remember the basic concepts of the Fourier Transform; I'm going to touch briefly on the inverse Fourier Transform.

5) Least deep and most incomplete explanation ever proposed on why the *inverse* Fourier Transform works.

Actually, scratch that, that's too much, let's do this…

5) A good way to remember what the inverse Fourier Transform looks like if you bought into Section 4.

The inverse Fourier Transform is the opposite of the Fourier Transform. If I'm the magic IFT oracle, you would give me a bunch of frequency/amplitude pairs – in other words, you're saying: here's how much of each frequency live in some signal. I would then tell you what the original signal was.

So this is the opposite of the Fourier Transform. With no regards to mathematical depth, what's the opposite of this?

$$X(\omega) = \int_{t=-\infty}^{t=\infty} x(t)e^{-i\omega t}dt$$

How about we just switch all the t's and ω's, and turn the inside of the integral upside-down:

$$x(t) = \int_{\omega=-\infty}^{\omega=\infty} X(\omega)e^{i\omega t}d\omega$$

This is very very close to the conventional representation for the IFT, which includes a normalization factor and looks like this:

$$x(t) = \frac{1}{2\pi}\int_{\omega=-\infty}^{\omega=\infty} X(\omega)e^{i\omega t}d\omega$$

I have been very careless with absolute values here, so you can just trust that the $1/2\pi$ is necessary just to make sure that if I put x(t) into a Fourier Transform and take the IFT of the result, I get x(t) back.

6) What did we leave out?

Let's see… aside from all mathematical depth and justification, the most important omission here is that my logic – which is really more like a pneumonic device – completely ignores phase and completely ignores the complex part of the FT. This is okay for two reasons… (1) this is just a basic intuition and a way of remembering what the FT looks like, and (2) many folks who are likely to read this document use the FT often just for computing power spectra (which is defined as the *magnitude* of the *real* part of the FFT of a signal). Phase? What phase. We don't need no stinking phase. Complex part? What complex part. We don't need no stinking complex part.

Also, to be clear, throughout this document I used notation consistent with the *continuous* Fourier Transform, but as far as understanding, recognizing, and remembering the basic formula, this all applies just fine to the discrete Fourier Transform (DFT and IDFT) and the discrete-time Fourier Transform (DTFT and IDTFT), whose typical formulas look very very much like the FT and IFT I've presented here.

Appendix A: Matlab code

```matlab
% Quick demonstration of using the Fourier transform in Matlab
% to measure the frequency content of a signal.
%
% Dan Morris, 2006
% http://cs.stanford.edu/~dmorris


% Number of points we'll have in our signal
N = 1024;


% The time of our "recording interval", abitrarily 10 seconds
T = 10;


% Define time a time axis
t = [0:N-1]/N;


% Convert time to seconds
t = t*T;


% Let's add together three sine waves with different frequencies and
% amplitudes
f1 = 2; a1 = 7;
f2 = 3; a2 = 11;
f3 = 5; a3 = 13;


% Define our test function
f = a1*sin(2*pi*f1*t) + a2*sin(2*pi*f2*t) + a3 * sin(2*pi*f3*t);


% Plot our test function and make it look pretty
```

```matlab
figure(1);
plot(t,f);
xl = xlabel('Time (s)');
set(xl,'FontSize',15);
yl = ylabel('Signal (made-up units)');
set(yl,'FontSize',15);
tl = title('Made-up signal for demonstrating fft');
set(tl,'FontSize',15);
set(gcf,'MenuBar','none');


%%

% Take the fourier transform of our signal, then take its magnitude,
% since for today, we're not interested in phase.  To convert to the
% amplitudes we provided in our original signal, divide by N/2; that's just
% the Matlab convention for the fft function.
p = abs(fft(f))/(N/2);

% We're only interested in positive frequencies for this demonstration, so
% only use the second half of the stuff we got back from fft().  The first
% half contained negative frequencies.
p = p(1:N/2);

% Find the corresponding frequency in Hz (we get frequencies back from FFT
% in radians)
freq = [0:N/2-1]/T;

% Plot our Fourier components and make them look pretty
figure(2);

% plot the power
plot(freq,p);
xl = xlabel('Time (s)');
set(xl,'FontSize',15);
yl = ylabel('Signal (made-up units)');
set(yl,'FontSize',15);
tl = title('Fourier components of our signal');
set(tl,'FontSize',15);
set(gcf,'MenuBar','none');

% Zoom in on an interesting range of the graph
xlim([0,10]);
```